A Lifting-Surface Solution for Vortex-Induced Airloads

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Planar lifting-surface theory is applied to the problem of the loads induced on an infinite aspect-ratio wing by a straight, infinite, free vortex in a subsonic, compressible, freestream; the vortex lies in a plane parallel to the plane of the wing, at a given distance below it, and at an arbitrary angle with the wing centerline. The solution is obtained in the form of an aerodynamic influence function for an infinite aspect-ratio wing in an oblique, sinusoidal gust. The lifting-surface theory kernel function for this wing is presented. This kernel includes as limits the linear, aerodynamic kernel functions for the problems of steady, three-dimensional flow and unsteady, two-dimensional flow, and for the problems of incompressible, two-dimensional flow and transonic, three-dimensional flow. An approximate analytic expression is given for the lift influence function, suitable for practical use in the routine calculation of vortex-induced airloads.

Nomenclature

b	=	wing semichord
G^A,G	=	normalized differential pressure (aerodynamic influence function)
$g_n{}^A$, g_L, g_M, g_C	=	aerodynamic influence functions
h	=	distance of vortex below wing
K^A,K	=	aerodynamic kernel functions
L^A,L	=	differential pressure on lifting surface (L also denotes section lift)
M	=	freestream Mach number
p	=	perturbation pressure
V	=	freestream velocity
w	=	downwash velocity
α	=	Prandtl-Glauert compressibility correction factor: $\alpha^2 = 1 - M^2$
β,B	=	elliptic and hyperbolic domain parameters: $\beta^2 = -B^2 = 1 - (M/\sin\Lambda)^2$
Λ	=	angle between vortex and wing
ν	=	wave number
ϕ	=	velocity potential
ψ	=	acceleration potential
$\overline{(\ )}$	=	Fourier transform

Coordinate systems

(x,y,z)	=	original coordinate system of model problem
(s',r',z)	=	coordinate system translating with vortex
(s^A,r^A,z)	=	coordinate system for loads
(s,r,z)	=	coordinate system for circulation

Introduction

In many operating conditions, a large contribution to the harmonic air-loading on a helicopter rotor blade is due to the passage of the blade very close to the strong tip vortex from a preceding blade. Although this source of harmonic airloading has long been recognized,[1] a practical method for accurate calculation of the loads induced by a nearby vortex has not been available. In the calculation of the aerodynamic loading on a helicopter rotor blade, it has been customary to use lifting-line theory.[2] It is assumed that the flow over the blade is locally two-dimensional, and the influence of the rest of the blade and the rotor wake is represented only by a uniform downwash at the blade section. Then two-dimensional, unsteady-airfoil theory (or experimental or empirical section-loads data) is used to obtain the section lift and moment. The accuracy of the airloads in current calculations is restricted by the use of lifting-line theory. It is a well established limitation that lifting-line theory is not valid for the large variations of the downwash along the span associated with a nearby vortex; moreover, this theoretical limitation is compounded by the practical difficulty of handling in sufficient detail the shed and trailed wake induced by this vortex. Therefore it is necessary to turn to the more accurate lifting-surface theory to obtain the vortex-induced airloading.

Lifting-surface theory is a well-developed aerodynamic tool.[3] It is however characterized by a large amount of calculation required to obtain the loading for a single case. The problem of the loads induced on a fixed wing by a free vortex has in fact been studied using a direct application of the standard lifting-surface theory techniques.[4,5] The extent of the calculations involved in these methods, as well as the difficulties encountered in applying them to the particular downwash distribution due to a free vortex, prohibit the direct application of the conventional lifting-surface theory techniques to the already highly iterative calculation of rotary-wing airloads. The proper procedure is to construct a sufficiently general model for the vortex-induced-airloads problem and to obtain the loads in this model using lifting-surface theory. Then the solution for this model problem may be routinely applied to the calculation of rotary-wing airloads.

The model used for the vortex-induced-airloads problem is an infinite aspect-ratio wing in a subsonic, compressible, freestream, and a straight, infinite vortex at an arbitrary

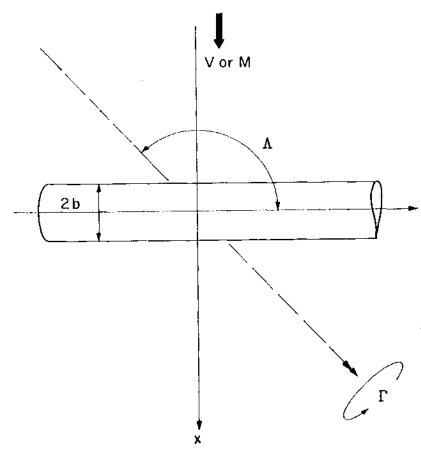

Fig. 1 Model problem for vortex-induced airloads.

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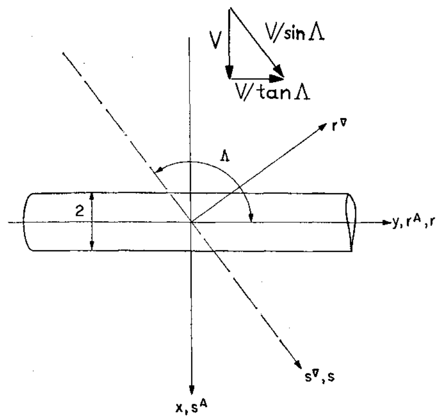

Fig. 2 The coordinate systems used: the (x,y) system is fixed; all the others are translating with the free vortex; the third axis for all systems is z, directed upward.

angle Λ with the wing (Fig. 1). The vortex is considered a straight line, thus the distortion of the vortex by the flowfield of the wing is an aspect of the probelm that is not treated here at all. The vortex lies in a plane parallel to the plane of the blade, at a distance h below it, and is convected past the blade by the freestream. Thus the vortex induced downwash in the plane of the wing is a one-dimensional downwash field, depending only on the perpendicular distance from the line that is the projection of the vortex on the wing plane. The model problem is generalized to include any such convected, one-dimensional downwash field. Only the range $\pi/2 < \Lambda < \pi$ will be considered; the results for $0 < \Lambda < \pi/2$ follow from symmetry. For $\Lambda = \pi/2$, the problem becomes the steady, three-dimensional flow due to a perpendicular to the wing. For $\Lambda = \pi$, it is the unsteady, two-dimensional flow of a point vortex past an airfoil. Some singular behavior of the problem may be expected, as the first of these limits is an elliptic problem, whereas the second is hyperbolic (due to the time dependence). The transition between the elliptic and hyperbolic problems will be shown to occur at $M = \sin\Lambda$.

Because of the infinite geometry of the model, the problem is steady in a coordinate system with its origin translating along the blade centerline, following the projection of the convected vortex. A natural coordinate system for such a translating frame of reference has one coordinate (s') aligned in the direction of the free vortex (Fig. 2). In this frame of reference the vortex is stationary, so the problem is steady and there is no shed vorticity (wake vorticity normal to the direction of the relative freestream, generated by time-varying blade circulation). Thus both the induced wake-vorticity behind the wing and the relative freestream velocity in this coordinate system must be in the direction of the free vortex (the s' direction). The relative freestream Mach number in this translating coordinate system is $M/\sin \Lambda$, and the wing is swept at an angle Λ to the relative freestream. The magnitude and direction of this relative freestream may also be found geometrically from the vector sum of the actual freestream (V) and the velocity of translation of the steady coordinate system ($V/\tan\Lambda$), as shown in Fig. 2. It is seen that the relative freestream attains a sonic value, $M/\sin\Lambda = 1$, at the transition between the elliptic and hyperbolic domains.

Linear lifting-surface theory is used to obtain the solution to the model problem in the form of an integral equation relating the pressure and downwash at the wing surface. This integral equation is a double integral over the wing surface; it is transformed into a single (chordwise) integral by the use of the Fourier transform with respect to the span variable. The kernel function in this integral equation is the general aerodynamic kernel for an infinite aspect ratio wing with the vorticity in the wake in one direction only, and no variation of the vorticity strength along that direction (these two conditions are actually equivalent due to continuity of vortex lines in the wake). For such a wing there is a coordinate system translating along its centerline in which there exists only trailed vorticity (vorticity in the direction of the relative freestream). Thus this kernel is more restricted than the general unsteady, three-dimensional kernel, which includes both trailed and shed wake-vorticity. The existence of such a steady coordinate system means that with the use of the Fourier transform there is one wave number which includes both the time and spanwise variables, in a particular combination specified by the geometric parameter Λ. For $\Lambda = \pi/2$, this kernel is a form of the kernel function for a steady three-dimensional, lifting surface (as, for example, in Ref. 6). For $\Lambda = \pi$, it is the kernel for the unsteady motion of a two-dimensional airfoil in compressible flow (Possio's kernel, as in Ref. 7).

The downwash in the integral equation must be such that it is steady in some frame of reference translating along the blade, so that the kernel function (the arrangement of the wake vorticity) previously described is applicable. Also, here is sought the response of the wing to disturbances caused by external agents in the flow, rather than by the motion of the wing itself, and so convected along with the flow. The appropriate downwash is like that due to a skewed free vortex; it is a one-dimensional (depending only on r'), convected downwash field. With the use of the Fourier transform this downwash field is replaced by a sinusoidal gust, at an angle Λ with the wing, with the crests aligned with the vortex direction (the s' axis). For $\Lambda = \pi/2$, the downwash is steady, with sinusoidal variation along the span of the wing. For $\Lambda = \pi$, the problem is a two-dimensional airfoil in a sinusoidal gust. The incompressible, two-dimensional case, $\Lambda = \pi$ and $M = 0$, may be solved in closed form using the standard results of thin airfoil theory; the solution for the lift for this case is the Sears function.[8]

The infinite, straight geometry of the wing and the vortex or downwash field is necessary in order for a steady frame of reference to exist for the problem. It also allows the introduction of the Fourier transform into the integral equation. By replacing the spanwise variable by the wave number (which appears simply as a parameter), it reduces the integral equation from a double integral to a single integral, a great simplification for numerical computation. The use of the Fourier transform also depends on the linearity of the problem, and in this respect it has greater significance. The consideration of only small disturbances leads to linear lifting-surface theory. For a linear system, the aerodynamic loads respond to the harmonic content of the downwash; that is, the physical significance of linearity is that a downwash of one wave number induces only loads of that same wave number. Thus, rather than obtain the loads due to a specific downwash, it is only necessary here to solve for the loads due to a sinusoidal gust. The solution is in the form of the airloading due to a harmonic input at a given wave number and of unit amplitude; it may be considered an aerodynamic influence function. The solution for a specific downwash is then obtained by the principle of superposition. The consideration of only harmonic excitation, rather than a specific downwash or even a family of downwash fields, is a standard technique in unsteady aerodynamics; the simplification due to this step is particularly important in the present problem where there are already two parameters of the model, M and Λ, and the introduction of other parameters from the downwash field would make it impossible to find a sufficiently general, yet practical solution. For a linear system the proper procedure is the indirect approach of first obtaining the general aerodynamic influence function. The consideration of harmonic excitation here in terms of a span wave number rather than a frequency points out a fact that tends to be neglected in unsteady aerodynamics: the influence function has little value in itself, but must be used with the principle of superposition to obtain the actual loading due to a particular (but also sufficiently general) family of downwash fields.

An arbitrary downwash field encountered by a wing may be represented as a superposition of one-dimensional, convected downwash fields over the entire range of angle Λ. This is equivalent to a double-Fourier-integral representation of the downwash, and thus the aerodynamic influence function for a sinusoidal gust is applicable to fully two-dimensional downwash fields. This is a spectral-analysis approach, and besides the application to discrete downwash fields, it is probably more useful applied to a turbulent gust. For this application, Filotas[9] arrived at the same model problem considered here, although he treated only the incompressible case. As his aims differed from those presented here, so also his means and results differed. These will be discussed further below.

Derivation of the Integral Equation

In what follows all quantities are nondimensional, based on the fluid density, the wing semichord, and the freestream velocity (ρ,b,V). The several coordinate systems used are shown in Fig. 2. The (s',r') system is the translating coordinate system, with s' aligned in the vortex direction. The (s^A,r^A) system is the one in which the problem must be solved to obtain the loads. One coordinate is along the span (so the Fourier transform may be used) and the other is in the chordwise direction (along which it is necessary to integrate to obtain the section airloading). The blade leading and trailing edges are given by $s^A = \pm 1$; this system is orthogonal. The (s,r) system is the one in which the problem must be solved to obtain the circulation. One coordinate is along the span (so the Fourier transform may be used) and the other is in the s' direction (along which it is necessary to integrate to obtain the circulation). The s metric has been stretched so that the blade leading and trailing edges are still given by $s = \pm 1$; this system is not orthogonal.

The solution is most conveniently formulated in terms of the acceleration potential ψ. The linearized equation of motion is

$$[\nabla^2 - M^2(\partial/\partial x + \partial/\partial t)^2]\psi = 0 \tag{1}$$

where

$$\psi = (\partial/\partial x + \partial/\partial t)\phi = -p \tag{2}$$

Here ϕ is the velocity potential and p is the perturbation pressure. The boundary conditions are

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = w \text{ on the airfoil} \tag{3}$$

$$\Delta\psi = -\Delta p = 0 \text{ off the airfoil} \tag{4}$$

where Δ means the difference between the quantities at $z = 0^+$ and at $z = 0^-$. Here the downwash w is a function of r' only. In the (s',r') system the equation of motion becomes

$$\{[1 - (M/\sin\Lambda)^2](\partial^2/\partial s'^2) + (\partial^2/\partial r'^2) + \partial^2/\partial z^2\}\psi = 0 \tag{5}$$

$$\psi = -p = (1/\sin\Lambda)\partial\phi/\partial s' \tag{6}$$

It is seen that in this system the problem is indeed steady, and elliptic or hyperbolic as $M/\sin\Lambda$ is less than or greater than one.

The elementary lifting solution for the acceleration potential is the dipole solution, denoted by ψ_d. Using superposition, the acceleration potential at an arbitrary point due to a lifting surface may be written

$$\psi(s^A,r^A,z) = \int_{-\infty}^{\infty}\int_{-1}^{1} L^A(\sigma^A,\rho^A)\psi_d(s_o',r_o',z)d\sigma^A d\rho^A \tag{7}$$

where $s_o' = s' - \sigma'$, $r_o' = r' - \rho'$, and $L^A(s^A,r^A)$ is the differential pressure across the lifting surface:

$$L^A(s^A,r^A) = -\Delta p = \Delta\psi \tag{8}$$

It is the advantage of the acceleration potential that the boundary condition off the airfoil, $\Delta\psi = -\Delta p = 0$, is automatically satisfied by placing the elementary solutions only on the lifting surface. Integration of Eq. (6) and application of the boundary condition on the airfoil gives

$$w(r') = \lim_{z\to 0}\frac{\partial}{\partial z}\int_0^{\infty}\psi|_{s_o = s_o - \lambda}\, d\lambda \tag{9}$$

where $s_o = s - \sigma$; the notation in the integrand means that ψ is expressed as a function of s by means of the coordinate transformations between the (s^A,r^A), (s',r') ,and (s,r) systems, and then the quantity s_o is replaced by the quantity $(s_o - \lambda)$. Substitution of ψ from Eq. (7) gives the integral equation:

$$w(r') = \int_{-\infty}^{\infty}\int_{-1}^{1}\frac{L^A(\sigma^A,\rho^A)}{2\pi/\alpha}\left\{\lim_{z\to 0}\frac{\partial}{\partial z}\int_0^{\infty}\psi_d|_{s_o = s_o - \lambda}\, d\lambda\,\frac{2\pi}{\alpha}\right\} d\sigma^A d\rho^A \tag{10}$$

It should be noted that here α denotes the Prandtl-Glauert compressibility factor: $\alpha^2 = 1 - M^2$. Writing $L^A(s^A,r^A)$ as a Fourier integral

$$L^A(s^A,r^A) = \int_{-\infty}^{\infty}\bar{L}^A(s^A,\nu)e^{i\nu r^A}d\nu \tag{11}$$

and taking the Fourier transform of the integral equation gives

$$\int_{-1}^{1}\bar{G}^A\left(\sigma^A, \frac{\nu}{\sin\Lambda}\right)K^A\left(s_o{}^A, \frac{\nu}{\sin\Lambda}\right)d\sigma^A = -e^{i(\nu/\sin\Lambda)s^A\cos\Lambda} \tag{12}$$

where $s_o{}^A = s^A - \sigma^A$. It is seen that the downwash now is just an oblique, sinusoidal gust, $e^{i(\nu/\sin\Lambda)s^A\cos\Lambda}$. The kernel function is

$$K^A(s_o{}^A,\nu/\sin\Lambda) = \left[-\frac{2\pi}{\alpha}\int_{-\infty}^{\infty} e^{-i\nu r_o{}^A}\lim_{z-0}\frac{\partial}{\partial z}\int_0^{\infty}\psi_d|_{s_o = s_o - \lambda}\, d\lambda dr_o{}^A\right] \tag{13}$$

and the aerodynamic influence function is

$$\bar{G}^A(s^A,\nu/\sin\Lambda) = \bar{L}^A(s^A,\nu/\sin\Lambda)/[(2\pi/\alpha)(1/\sin\Lambda)\bar{w}(\nu/\sin\Lambda)] \tag{14}$$

where $1/\sin\Lambda\bar{w}(\nu/\sin\Lambda)$ is the Fourier transform of $w(r')$ with respect to r^A, at $s^A = 0$. The wave number has been written in the form $\nu/\sin\Lambda$; this quantity has the limits of a span wave number for $\Lambda = \pi/2$, and of a reduced frequency for $\Lambda = \pi$. It is in fact the wave number of the sinusoidal gust.

The integral equation may be also obtained in the (s,r) system, as

$$\int_{-1}^{1}\bar{G}(\sigma,\nu/\sin\Lambda)K(s_o,\nu/\sin\Lambda)d\sigma = -1 \tag{15}$$

where $s_o = s - \sigma$. The aerodynamic influence function is

$$\bar{G}(s,\nu/\sin\Lambda) = \bar{L}(s,\nu/\sin\Lambda)/[(2\pi/\alpha)(1/\sin\Lambda)\bar{w}(\nu/\sin\Lambda)] \tag{16}$$

where $\bar{L}(s,\nu/\sin\Lambda)$ is the Fourier transform with respect to r of the loading $L(s,r) = -\Delta p$. The kernel is given in terms of K^A by

$$K(s_o,\nu/\sin\Lambda) = e^{-i(\nu/\sin\Lambda)s_o\cos\Lambda}K^A(s_o{}^A = s_o,\nu/\sin\Lambda) \tag{17}$$

a result that follows simply from the properties of Fourier transforms, the coordinate transformation between the (s,r) and (s^A,r^A) systems [$s_o = s_o^A$ and $r_o \sin\Lambda = r_o^A \sin\Lambda + s_o^A \cos\Lambda$; recall from the discussion of the coordinate systems that the (s,r) system is not orthogonal, and furthermore that the s metric has been stretched by a factor of $1/\sin\Lambda$], and the fact that the kernels are themselves Fourier transforms (K with respect to r_o and K^A with respect to r_o^A).

The kernels depend on the parameters M and Λ; specifically they depend on the sign of the quantity

$$\beta^2 = -B^2 = 1 - (M/\sin\Lambda)^2 \tag{18}$$

that is, on whether the flow is in the elliptic or hyperbolic domain. The derivation of the kernel functions will not be given here, since it follows the standard techniques of aerodynamic theory; in particular it is very similar to Possio's method.[7] The detailed derivation of the kernels may be found in Ref. 10.

Elliptic Kernel

The elliptic domain is given by $M < \sin\Lambda$, or $\beta^2 > 0$. The elliptic kernel may be written

$$K_\beta{}^A(s_o{}^A,\nu/\sin\Lambda) = e^{i(\nu/\sin\Lambda)s_o{}^A\cos\Lambda}\Big\{e^{ia\mu s_o{}^A}\Big[\pm aK_1(a|s_o{}^A|) + ia\mu K_o(a|s_o{}^A|)\Big] - \frac{i}{2}\frac{1}{\alpha}\frac{\nu}{\sin\Lambda}\Big[\pi i + \ln\frac{\alpha - \cos\Lambda}{\alpha + \cos\Lambda}\Big] \pm \frac{1}{\alpha}\frac{\nu}{\sin\Lambda}\Big[1 - \Big(\frac{\cos\Lambda}{\alpha}\Big)^2\Big]^{-1/2}\int_0^{a|s_o{}^A|} e^{\pm i\mu\xi}\, K_o(\xi)\partial\xi\Big\} \tag{19}$$

where

$$\pm = s_o{}^A|s_o{}^A| \tag{20a}$$

$$a = (1/\alpha)\nu/\sin\Lambda[1 - (\cos\Lambda/\alpha)^2]^{1/2} \tag{20b}$$

$$\mu = -\cos\Lambda/\alpha[1 - (\cos\Lambda/\alpha)^2]^{-1/2} \tag{20c}$$

$$a\mu = (1/\alpha)(\nu/\sin\Lambda)(-\cos\Lambda/\alpha) \tag{20d}$$

and K_0, K_1 are modified Bessel functions. For $\Lambda = \pi/2$, the kernel may be written as

$$K_\beta{}^A(s_o{}^A,\nu/\sin\Lambda) = -\frac{1}{2\alpha}\int_{-\infty}^{\infty} e^{-i\nu r_o{}^A}\frac{1}{r_o{}^{A^2}} \times \left[1 + \frac{s_o{}^A}{(s_o{}^{A^2} + \alpha^2 r_o{}^{A^2})^{1/2}}\right] dr_o{}^A \tag{21}$$

which may be identified as the Fourier transform of the steady, three-dimensional, lifting-surface kernel.

Hyperbolic Kernel

The hyperbolic domain is given by $M > \sin\Lambda$, or $B^2 > 0$. The hyperbolic kernel may be written

$$K_B{}^A(s_o{}^A,\nu/\sin\Lambda) = e^{i(\nu/\sin\Lambda)s_o{}^A\cos\Lambda}\Big\{\frac{\pi i}{2}e^{ia\mu s_o{}^A} \times [\mp aH_1{}^{(2)}(a|s_o{}^A|) - ia\mu H_o{}^{(2)}(a|s_o{}^A|)] - \frac{i}{2}\frac{1}{\alpha}\frac{\nu}{\sin\Lambda} \times \ln\frac{-\cos\Lambda + \alpha}{-\cos\Lambda - \alpha} \mp \frac{\pi i}{2}\frac{1}{\alpha}\frac{\nu}{\sin\Lambda}\Big[\Big(\frac{\cos\Lambda}{\alpha}\Big)^2 - 1\Big]^{-1/2} \times \int_o^{a|s_o{}^A|} e^{\pm i\mu\xi}H_o{}^{(2)}(\xi)d\xi \tag{22}$$

where

$$\pm = s_o{}^A/|s_o{}^A| \tag{23a}$$

$$a = (1/\alpha)(\nu/\sin\Lambda)[(\cos\Lambda/\alpha)^2 - 1]^{1/2} \tag{23b}$$

$$\mu = (-\cos\Lambda/\alpha)[(\cos\Lambda/\alpha)^2 - 1]^{-1/2} \tag{23c}$$

$$a\mu = (1/\alpha)(\nu/\sin\Lambda)\,(-\cos\Lambda/\alpha) \tag{23d}$$

and $H_o{}^{(2)}$, $H_1{}^{(2)}$ are Hankel functions. For $\Lambda = \pi$, and writing $\nu/\sin\Lambda = k$ (a reduced frequency), the kernel function may be identified as Possio's form of the kernel for unsteady, compressible flow about a two-dimensional thin airfoil.[7] Also then for this limit the compressed span variable $r^A \sin\Lambda$ should be interpreted as the time variable.

The kernels in the two domains are actually fundamentally related, and one may be obtained from the other by noting that $\beta = iB$.

For $\nu/\sin\Lambda = 0$ the kernels reduce to

$$K^A(s_o{}^A,0) = K(s_o,0) = 1/s_o{}^A \tag{24}$$

For this kernel the integral equation inverts directly to give

$$\bar{G}^A(s^A, 0) = -(1/\pi)[(1 - s^A)/(1 + s^A)]^{1/2} \tag{25}$$

The limit $\nu/\sin\Lambda \rightarrow 0$ corresponds to a wavelength of the sinusoidal gust very large compared with the wing chord. The problem reduces to a two-dimensional airfoil in a uniform downwash, and Eq. (25) is the standard result from thin-airfoil theory.

Transitional Kernel

The transitional case is $M = \sin\Lambda$ or $\beta = B = 0$. The equation of motion (5) then becomes

$$[(\partial^2/\partial r'^2) + (\partial^2/\partial z^2)]\psi = 0 \tag{26}$$

which is simply Laplace's equation in two dimensions. The reduction in dimensions in the equation of motion indicates a violation of the linearity assumption. To examine the nature of this case, it is necessary to return to the exact equation for the velocity potential in three-dimensional, unsteady, compressible flow. The equation for the first order potential when $M \cong \sin\Lambda$ may be obtained, and it is of exactly the same form as the equation for the potential in three-dimensional, steady, transonic flow; indeed, for $\Lambda = \pi/2$, that is exactly what the transitional case becomes. For $\Lambda > \pi/2$ this is not a transonic flow, of course; there is no region where the flow has sonic velocity as long as M is less than one. There is however an important velocity that has become sonic, namely the vector sum of the Mach number and the velocity of translation of the vortex along the blade centerline; this is just the relative velocity in the (s',r') frame of reference, so the transitional case is given by $M/\sin\Lambda = 1$. It is this combination of the physical and geometric velocities at the sonic velocity, acting as a disturbance reinforcement process, that makes the linear assumption not valid. Because of the nonlinearity of the correct equation of motion for the transitional case (the transonic flow equation), it is not possible to obtain an aerodynamic influence function for this case, for the existence of such a function is a property of a linear system only. Properly solved, the transitional solution would have to be obtained for every particular downwash distribution, and even if these solutions could be obtained they would be of little practical use in rotary-wing airloads calculations.

The two-dimensional quality of the linearized problem is due to the coincidence of the sum of the freestream and vortex-translation velocities with sonic speed; disturbances produced by the downwash distribution remain exactly in step with it as both are convected along the blade. Of course, nonlinearities destroy the exact two-dimensional nature of the problem, but the two-dimensional problem does roughly represent the physical character of the transitional case. Moreover, as a linear problem it does have a solution in the form of an aerodynamic influence function. Finally, it is at least a valid limit of the linear elliptic and hyperbolic results. For these reasons, only the linearized transitional case is considered here.

The linear transitional kernel may be obtained as a limit of the elliptic or hyperbolic kernels. The result is

$$K_T{}^A(s_o{}^A,\nu/\sin\Lambda) = e^{-i(\nu/\sin\Lambda)s_o{}^A\alpha}\left\{\frac{e^{i(1/\alpha)(\nu/\sin\Lambda)s_o{}^A}}{s_o{}^A} - i\frac{1}{\alpha}\frac{\nu}{\sin\Lambda}\left[\gamma + \frac{i\pi}{2} + \ln\left(\frac{1}{\alpha}\frac{\nu}{\sin\Lambda}|s_o{}^A|\right)\right] - i\frac{1}{\alpha}\frac{\nu}{\sin\Lambda} \times \int_o^{\frac{1}{\alpha}\frac{\nu}{\sin\Lambda}|s_o{}^A|}\frac{e^{\pm it}-1}{t}\,dt\right\} \quad (27)$$

It may be seen that the two-dimensional, incompressible problem, $\Lambda = \pi$ and $M = 0$, is a special case of the transitional problem. An examination of the orders of the terms in the nonlinear transitional equation gives the conditions for linearization to be valid; it is found that the actual transitional domain (the region in the vicinity of the line $M = \sin\Lambda$ where nonlinear effects are important) is a narrow strip with a width proportional to the Mach number squared. Thus the linearization of the equation of motion in the two-dimensional, incompressible case is uniformly valid. The very special nature of incompressible, two-dimensional flow is well known; here it is seen that the exceptional character of the flow arises because it is a particular limit of the transitional case, which is itself a singular limit of the general airfoil problem considered here. For this case alone the integral equation may be solved in closed form by classical techniques. The loading on a two-dimensional airfoil due to a sinusoidal gust in an incompressible flow is the Sears function[8]; the influence function is

$$\bar{G}^A(s^A,\nu/\sin\Lambda) = \frac{1}{\pi}\left(\frac{1-s^A}{1+s^A}\right)^{1/2} \times [\bar{\Gamma}_o(\nu/\sin\Lambda)C(\nu/\sin\Lambda) - iJ_1(\nu/\sin\Lambda)] \quad (28)$$

where

$$\bar{\Gamma}_o(\nu/\sin\Lambda) = -J_0(\nu/\sin\Lambda) + iJ_1(\nu/\sin\Lambda) \quad (29)$$

$$C(\nu/\sin\Lambda) = H_1{}^{(2)}(\nu/\sin\Lambda)/[H_1{}^{(2)}(\nu/\sin\Lambda) + iH_0{}^{(2)}(\nu/\sin\Lambda)] \quad (30)$$

and J_0, J_1 are Bessel functions. The lift acts exactly at the quarter chord for all wave numbers, a result that is not true for any other values of M and Λ.

The important character of the model problem is that it includes as limits in M and Λ several types of flow about a lifting wing. The transitional region is an essential feature of this character, for it not only separates the elliptic and hyperbolic domains, but also joins transonic, three-dimensional and incompressible, two-dimensional flows. It would be expected that such a transition would be nonlinear. The disturbance-amplification character of the transitional domain will not actually be very important for the calculation of vortex-induced airloads on a rotary wing, because of the small probability of the coincidence of M and Λ at the transitional case simultaneously with the occurrence of a substantial peak in the vortex-induced downwash. It should be of more importance for application of the results to turbulence loads, where there will be some excitation at all Λ and $\nu/\sin\Lambda$.

Aerodynamic Influence Functions

The influence function $\bar{G}^A$ is written as a Glauert series:

$$\bar{G}^A(s^A,\nu/\sin\Lambda) = \sum_{n=0}^{\infty}\bar{g}_n{}^A(\nu/\sin\Lambda)f_n(s^A) \quad (31)$$

where

$$f_n(s^A) = \begin{cases}\tan\theta/2 & n = 0\\ \sin(n\theta) & n \geq 1\end{cases} \quad s^A = \cos\theta \quad (32)$$

and a similar expression is used for $\bar{G}(s,\nu/\sin\Lambda)$. The section lift, moment, and circulation may then be obtained by integration over the airfoil chord. The aerodynamic influence functions for the section lift, moment about the quarter chord, and circulation are

$$\bar{g}_L(\nu/\sin\Lambda) = \frac{\bar{L}(\nu/\sin\Lambda)}{(2\pi/\alpha)(1/\sin\Lambda)\bar{w}(\nu/\sin\Lambda)} = \pi[\bar{g}_0{}^A(\nu/\sin\Lambda) + \tfrac{1}{2}\bar{g}_1{}^A(\nu/\sin\Lambda)] \quad (33)$$

$$\bar{g}_M(\nu/\sin\Lambda) = \frac{\bar{M}_{QC}(\nu/\sin\Lambda)}{(2\pi/\alpha)(1/\sin\Lambda)\bar{w}(\nu/\sin\Lambda)} = -\frac{\pi}{4}[\bar{g}_1{}^A(\nu/\sin\Lambda) + \bar{g}_2{}^A(\nu/\sin\Lambda)] \quad (34)$$

$$\bar{g}_c(\nu/\sin\Lambda) = \frac{\bar{\Gamma}(\nu/\sin\Lambda)}{(2\pi/\alpha)(1/\sin\Lambda)\bar{w}(\nu/\sin\Lambda)} = \pi[\bar{g}_0(\nu/\sin\Lambda) + \tfrac{1}{2}\bar{g}_1(\nu/\sin\Lambda)]e^{i\frac{\nu}{\sin\Lambda}\cos\Lambda} \quad (35)$$

All these are Fourier transforms with respect to r^A: $2\pi/\alpha$ is the lift curve slope at $\nu/\sin\Lambda = 0$, and $1/\sin\Lambda\ \bar{w}(\nu/\sin\Lambda)$ is the Fourier transform of the downwash $w(r')$ with respect to r^A, at $s^A = 0$.

For all values of M and Λ except the incompressible, two-dimensional case, the integral equation must be solved numerically for the influence functions. However, for routine application of the influence functions, numerical results are not very useful, particularly with the results in terms of wave number. Thus the exact numerical solutions must be used to produce approximate analytic expressions for the influence functions. The requirements for the approximate expressions are that they accurately represent the behavior of the magnitude and phase over the required range of wave number, and furthermore that they be in a form convenient to use as Fourier transforms. The downwash due to a free vortex a distance h below the wing is

$$w(r^A\sin\Lambda) = (\Gamma_\infty/2\pi)(-r^A\sin\Lambda)/[(r^A\sin\Lambda)^2 + h^2] \quad (36)$$

and the Fourier transform of this is

$$\frac{1}{\sin\Lambda}\bar{w}(\nu/\sin\Lambda) = \frac{\Gamma_\infty}{2\pi}\frac{1}{\sin\Lambda}\frac{i\nu/\sin\Lambda}{2|\nu/\sin\Lambda|}e^{-h|\nu/\sin\Lambda|} \quad (37)$$

It is seen that for such a downwash distribution the high wave number behavior of the influence function is not important; for a minimum vortex distance of the order of 10% of the chord, it is sufficient to have the influence functions accurate out to about $\nu/\sin\Lambda = 6.0$, and this is the range that was chosen for the approximate expressions. An exponential function is very convenient as a Fourier transform [as in Eq. Eq. (37)], and whereas such a function must at large wave number underestimate the true influence function (which has only algebraic decay for large wave number), a sum of exponential terms can accurately represent the magnitude out to any finite wave number. Thus the expression constructed for the lift influence function was

$$\bar{g}_L(\nu/\sin\Lambda) = \left[-a_oe^{-c_o|\nu/\sin\Lambda|} + ia_o'\sin(b_o\nu/\sin\Lambda) \times e^{-c_o'|\nu/\sin\Lambda|} - e^{i\left(b_1\nu/\sin\Lambda - b_2\frac{\nu/\sin\Lambda}{|\nu/\sin\Lambda|}\right)} \times \sum_{m=1}^{2}a_m(i\nu/\sin\Lambda)^{2m}e^{-c_m|\nu/\sin\Lambda|}\right] \quad (38)$$

This expression is accurate for $\nu/\sin\Lambda$ out to about 6.0, and for M up to about 0.9. The constants (a_m, b_m, c_m) are functions of M and Λ, and were evaluated from the numerical solutions; expressions for these constants are given in the Appendix. Similar approximations for the moment, circulation, and pressure influence functions may be found in Ref. 10.

A comparison of the numerical and approximate solutions is shown in Fig. 3, for a typical case $M = 0$, $\Lambda = 135°$. Also shown is the result of an approximate solution obtained by Filotas.[9] In the present notation, his expression for the lift influence function is

$$\bar{g}_L(\nu/\sin\Lambda) = -\exp\left\{i\frac{\nu}{\sin\Lambda}\left[\cos\Lambda + \frac{\pi\left(\Lambda - \frac{\pi}{2}\right)\left(1 + \frac{1}{2}\sin\Lambda\right)}{1 + 2\pi\frac{\nu}{\sin\Lambda}\left(1 + \frac{1}{2}\sin\Lambda\right)}\right]\right\}\left[1 + \pi\frac{\nu}{\sin\Lambda}\left(1 + \cos^2\Lambda + \pi\frac{\nu}{\sin\Lambda}\sin\Lambda\right)\right]^{-1/2} \quad (39)$$

Equation (39) represents the phase very well, while Eq. (38) approximates the magnitude more closely over the important range of wave number. Equation (39) is useful for working with spectra or isolated numerical calculations, but for routine application to the calculation of vortex-induced loads an expression is required such as Eq. (38), which may be inverted to obtain the actual spanwise loading.

Vortex-Induced Airloading

As an example of the application of the approximate influence function, Eq. (38), it will be used with the vortex induced downwash [Eqs. (36) and (37)]. The section lift is obtained by use of the Fourier integral as

$$\frac{L(r^A\sin\Lambda)}{\frac{2\pi}{\alpha}\rho V^2 b} = \frac{\Gamma_\infty}{2\pi V b}\Big\{-a_o(-r^A\sin\Lambda)/[(r^A\sin\Lambda)^2 + (h + c_o)^2] + a_o'(-b_o)[-(r^A\sin\Lambda)^2 + (h + c_o')^2 + b_o^2]/\{[-(r^A\sin\Lambda)^2 + (h + c_o')^2 + b_o^2]^2 + 4(r^A\sin\Lambda)^2 \times (h + c_o')^2\} - \sum_{m=1}^{2} a_m\left(\frac{d}{dr^A\sin\Lambda}\right)^{2m}[-(r^A\sin\Lambda + b_1)\cos b_2 + (h + c_m)\sin b_2]/[(r^A\sin + b_1)^2 + (h + c_m)^2]\Big\} \quad (40)$$

Typical results from Eq. (40) are shown in Figs. 4 and 5, which give the magnitude of the peak section lift as a function of the vortex height h, for $M = 0$, $\Lambda = \pi/2$ and π, respectively. Figure 5 also shows the results of other calculations.[4,5,11] Kfoury and Silver used conventional lifting-surface techniques, and Simons used a lifting-line theory. From a comparison of the numerical and approximate influence functions for this case, the present solution should be valid down to about $h/b = 0.25$. Extensive results from Eq. (40) for the entire ranges of M, Λ, and h may be found in Ref. 10. With a closed form solution available for the induced loads in a general wing/vortex configuration, the application of lifting-surface theory results to the calculation of rotary-wing airloads becomes quite practical.

Conclusion

A model problem for the calculation of vortex-induced airloads has led to the examination of the nature of the lifting-surface theory solution for an infinite aspect-ratio wing in an oblique, sinusoidal gust. This problem encompasses and shows the nature of the transition between several familiar types of flow for a lifting airfoil: steady, three-dimensional flow and unsteady, two-dimensional flow; and two-dimensional, incompressible flow and three-dimensional, transonic flow. The broad scope of the problem of a wing in a compressible, oblique gust should make the solution a valuable tool for the calculation of rotary-wing airloads,[12] as well as for applications to such problems as aerodynamic noise[13] and loading due to turbulence.

Appendix: Approximate Lift Influence Function

Numerical solutions for the aerodynamic influence functions were obtained for 37 cases covering the entire ranges of M and Λ. From these the constants in the approximate lift influence function [Eq. (38)] were evaluated, and then expressions for the constants in terms of M and Λ were obtained. Because the nature of the kernel function depends on whether the case is in the elliptic, transitional, or hyperbolic domains, the expressions for the constants are different for these regions; the notation for the three domains is

ELP: $M < \sin\Lambda$

TRN: $M = \sin\Lambda$

HYP: $M > \sin\Lambda$

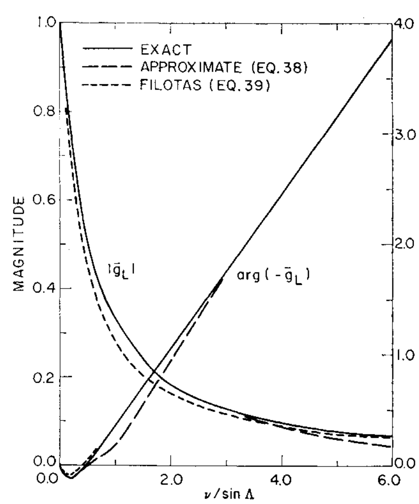

Fig. 3 Comparison of the exact and approximate lift influence functions: $M = 0$ and $\Lambda = 135°$.

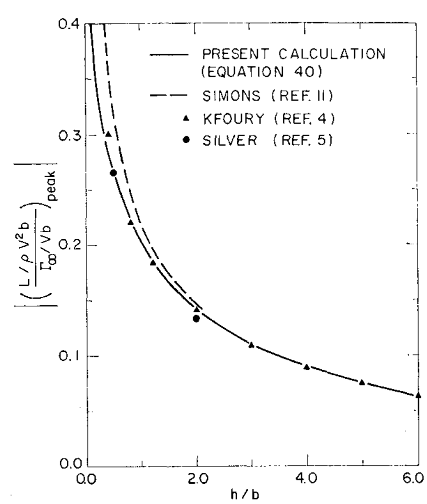

Fig. 4 Peak section lift as a function of vortex height: $M = 0$ and $\Lambda = 90°$.

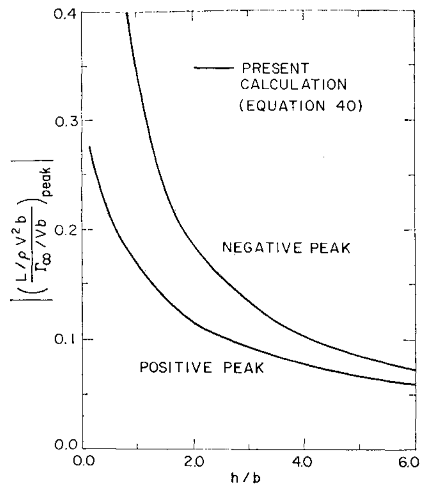

Fig. 5 Peak section lift as a function of vortex height: $M = 0$ and $\Lambda = 180°$.

The constants near the transitional domain are obtained from the following equation:

$$f = f_{\mathrm{TRN}}\, e^{-4\left(\frac{\Delta}{0.05}\right)^2} + f_{\mathrm{HYP\ or\ ELP}}\left[1 - e^{-4\left(\frac{\Delta}{0.05}\right)^2}\right]$$

where $\Delta = 1 - M/\sin\Lambda$, f is any of the constants a_m, b_m, c_m, and f_{TRN} is based on the same M as f_{HYP} or f_{ELP}. This corresponds to a transitional region with a width of the order of that indicated by the nonlinear equation. The expressions for the constants are given below (it should be noted that $\pi/2 < \sin^{-1}M < \pi$ to be consistent with the range of Λ):

$b_o = 5.12 + 1.88[(\Lambda - \sin^{-1}M)]/[\pi/2 - \sin^{-1}M]$, ELP
$b_o = 5.12$, TRN, $b_o = 5.12 - 4.12M^2$, HYP
$b_1 = -\cos\Lambda$, ELP, $b_1 = 1 - 0.7M^3$, TRN
$b_1 = e^{-10M} + 0.8(1 - 0.1\sin\Lambda)(1 - e^{-10M})$, HYP
$b_2 = \alpha/2(\Lambda - \pi/2)$, ELP

$$b_2 = \left\{\left[\frac{\alpha}{2}\left(\Lambda - \frac{\pi}{2}\right)\right]^2 + 0.2M + M^2 - \sin^2\Lambda\right\}^{1/2},$$

HYP or TRN

$a_o = 1$, $a_1 = -[0.434 + 1.09(1 - \sin\Lambda)^{0.94} - 0.607(1 - \sin\Lambda)^{2.46}](1 + 0.33M^{1.9})$, ELP
$a_1 = -0.917(1 - 0.52M^{5.38})$, HYP or TRN
$a_2 = [0.0084 + 0.0069(-\cos\Lambda)^{1.8}](1 - 0.26M^2)$, ELP
$a_2 = [0.0153 + 0.0188M - 0.0188M^2]$, HYP or TRN
$a_o' = [0.554(-\cos\Lambda) + 0.07\sin 2\Lambda](1 + 2.13M^{2.1})$, ELP
$a_o' = 0.554(1 + 0.34M^{1.9})$, TRN
$a_o' = 0.554(1 + 4.8M^{3.1})$, HYP
$c_o = [1.683 + 0.27(1 - \sin\Lambda)^{0.9} - 0.154(1 - \sin\Lambda)^{2.9}](1 + 0.72M^{2.3})$, ELP
$c_o = 1.799(1 + 0.6M^{2.6})$, TRN
$c_o = 1.799(1 + 1.84M^{3.46})(-\cos\Lambda)^{0.35}$, HYP
$c_1 = [1.417 + 0.366(1 - \sin\Lambda)^{0.84} - 0.392(1 - \sin\Lambda)^{2.0}](1 + 0.26M^{2.2})$, ELP
$c_1 = 1.391(1 + 0.47M^{2.5})$, TRN
$c_1 = 1.391(1 + 0.3M^{2.1})(-\cos\Lambda)$, HYP
$c_2 = [0.91 + 0.93(1 - \sin\Lambda)^{1.0} - 1.025(1 - \sin\Lambda)^{1.45}](1 + 0.07M^{2.5})$, ELP
$c_2 = 0.815(1 + 0.28M^{0.9})$, TRN
$c_2 = 0.815(1 + 0.55M^{0.5})(-\cos\Lambda)^{2.2}$, HYP
$c_o' = 5.9 + 1.5M$

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